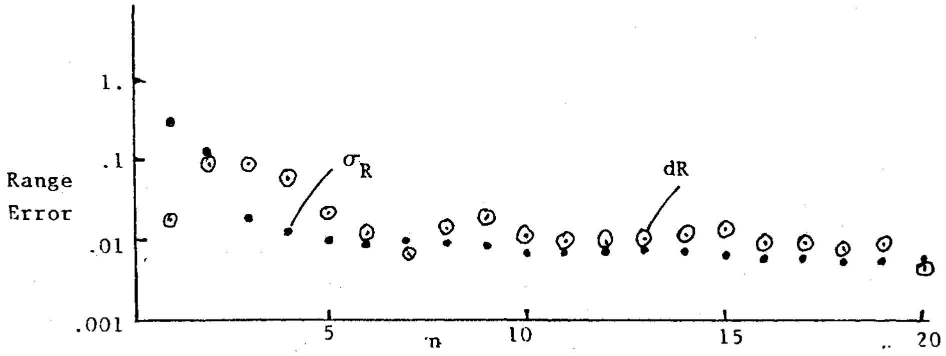

Fig. 2 Range estimation error and standard deviation.

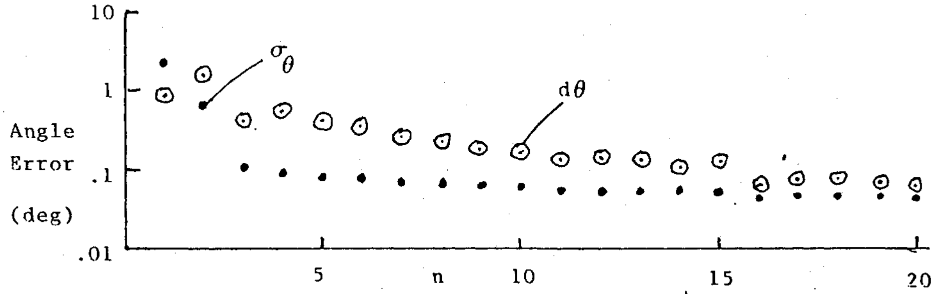

Fig. 3 Azimuth estimation error and standard deviation.

beam cycle, therefore A varies by 72 deg between data. The initial estimates $(\hat{r}, \hat{\theta})$ converge toward the actual values. The noise in the data is simulated as having a constant covariance, which may not be the case . If a dependence of noise on range can be estimated or conjectured, the effects of such noise on the estimate can be forecast.

Figures 2 and 3 show the time variations of errors in the estimates of range and azimuth angle, with their rms errors (1σ). Despite large nonlinearities and an indifferent initial estimate, the convergence is rapid and the errors $(dr, d\theta)$ are consistent with the estimated standard deviations. In an actual application, of course, the errors are not known and the standard deviations are found, as in this example.

Conclusions

Filter techniques have been applied to an estimation problem for a nonlinear, time-varying parameter system. The process yields estimates and rms errors that are consistent and well-behaved functions of the noisy data. The data in the example have been chosen to highlight the filter characteristics and do not apply to any current physical system.

Dominance of Stiffening Effects for Rotating Flexible Beams

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Introduction

SEVERAL proposed approaches to the derivation of the equations of motion for rotating flexible beams are essentially similar in the linear terms of the generalized coordinates associated with the elastic motion. All methods include the two opposing effects often referred to as *geometric stiffening* and *centrifugal softening*.[1] (Since this nomenclature is not accepted universally, we shall refer to these effects as simply stiffening and softening.) The softening effect, being purely the result of kinematics in a rotating frame, is the easiest to identify.

The stiffening effect is somewhat more complicated. Reference 1 shows that the stiffening can be interpreted as being the result of a quasisteady longitudinal strain. Reference 2 shows that the stiffening can be derived from consideration of an effect known as *foreshortening*. Other approaches[3,4] introduce an angular-velocity dependent component to the potential energy. All of these efforts are motivated by the desire to retain the computational simplicity of the Euler-Bernoulli beam theory. Other authors, including most recently Hanagud and Sarkar,[5] insist that the centrifugal stiffening effect can (and should) be derived from the nonlinear beam theory.

We show here that, for a class of beams rotating in a plane, the stiffening effect always dominates the softening effect. Although it seems intuitively obvious that a flexible beam rotating in a plane will always stiffen, a proof of this has not yet been presented. It is expected that this proof will also serve as a vehicle for further results.

Proof that Geometrical Stiffening Dominates Centrifugal Softening

The results presented are valid for any boundary conditions. For the case of a free-free beam, the flexible motion is described with respect to a rotating reference frame whose origin coincides with the center of mass of the beam. For the pinned-free and fixed-free cases, the origin of the rotating reference frame coincides with the pinned or fixed end of the beam. In both cases, we assume that there is no external axial force on the beam.

Consider a beam of arbitrary mass and stiffness distributions whose elastic axis and centroidal axis coincide and assume the constraint used to determine the angular velocity ω of the reference frame is not affected by the modal coordinates associated with elastic deformation. Let the total displacement of any point on the beam be $\boldsymbol{r}(x,t) = \hat{\boldsymbol{r}}(t) + x\boldsymbol{i} + u(x,t)\boldsymbol{j}$, and let the angular velocity be $\boldsymbol{\omega}(t) = \omega(t)\boldsymbol{k}$ (Fig. 1), where $\hat{\boldsymbol{r}}(t)$ locates the center of mass for the free-free case. The elastic

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displacement $u(x,t)$ perpendicular to the axis of the undeformed centroidal (reference) axis of the beam can be represented as

$$u(x,t) = \sum_{i=1}^{\infty} \phi_i(x) q_i(t)$$

where $\phi_i(x)$ are the eigenfunctions associated with the linear time-invariant (nonrotating) model and $q_i(t)$ the generalized coordinates. Note that because the rotation of the beam does not change the boundary conditions, the eigenfunctions of the nonrotating beam constitute a complete set of expansion functions.

Retaining only linear terms in the generalized coordinates $q_i(t)$, the equations for the flexible motion of such a structure can be shown to have the form

$$\ddot{q}_r(t) + (\lambda_r - \omega^2) q_r(t) + \omega^2 \sum_{i=1}^{\infty} d_{rs} q_s(t) = p_r(t), \quad r = 1,2, \ldots \tag{1}$$

where λ_r are the eigenvalues of the linear model, $p_r(t)$ the generalized forces, and

$$d_{rs} = \int_L^U m(x) x \left(\int_0^x \frac{\mathrm{d}\phi_r}{\mathrm{d}\zeta} \frac{\mathrm{d}\phi_s}{\mathrm{d}\zeta} \, \mathrm{d}\zeta \right) \mathrm{d}x$$

$$= - \int_L^U \left(\int_x^U m(\zeta) \zeta \, \mathrm{d}\zeta \right) \frac{\mathrm{d}\phi_r}{\mathrm{d}x} \frac{\mathrm{d}\phi_s}{\mathrm{d}x} \, \mathrm{d}x \tag{2}$$

are due to the centrifugal stiffening.[6,7] For the pinned-free and fixed-free cases, the lower limit on the integration is clearly $L = 0$; for the free-free case, the origin of the floating frame $x = 0$ locates the center of mass of the beam and the lower and upper limits, L and U, are determined from the center-of-mass expression. The softening effect is due to the term $\omega^2 q_r(t)$. The terms d_{rs} here may be compared with the terms ν_{ij} in Eq. (81) of Ref. 2 or the terms G_{ij} in Eq. (59) of Ref. 1. The equivalence of the two expressions for d_{rs} can be shown by integrating by parts and using the boundary conditions.

Equation (1) can be expressed in matrix form as

$$\ddot{q} + \Lambda q + \omega^2([D] - [I]) q = f \tag{3}$$

where $[D]_{rs} = d_{rs}$ $(r,s = 3,4,\ldots)$.

The answer to the question of whether centrifigual stiffening always dominates the geometrical softening can now be simply stated as follows: If the matrix $[D] - [I]$ is positive definite then centrifugal stiffening always dominates the geometric softening.

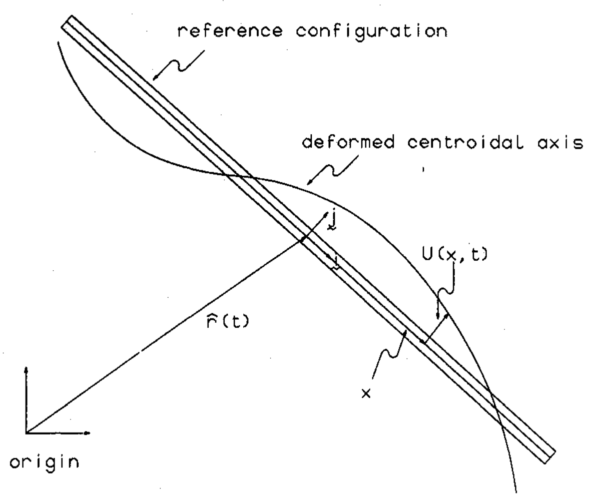

Fig. 1 Reference axes and deformation of a free-free beam.

We need to examine the conditions under which this is the case. First, consider the Taylor series expansion for the vector of eigenfunctions $\Phi = [\phi_1(x), \phi_2(x), \phi_3(x), \ldots]^T$

$$\Phi(x) = \sum_{i=1}^{\infty} \frac{\Phi^{(i)}(0)}{i!} x^i \tag{4}$$

The orthogonality condition is

$$\int_L^U m(x) \Phi(x) \Phi(x)^T \, \mathrm{d}x = [I] \tag{5}$$

Substituting Eqs. (4) and (5) into the expression for $[D] - [I]$ and integrating by parts, we get

$$[D] - [I] = - \left[\int_x^U m(\zeta) \zeta \, \mathrm{d}\zeta \right] \times \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \frac{1}{i!j!} \Phi^{(i)}(0) \Phi^{(j)}(0)^T ij \left. \frac{x^{i+j-1}}{i+j-1} \right|_{x=L}^{U} + \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \frac{1}{i!j!} \Phi^{(i)}(0) \Phi^{(j)}(0)^T \frac{ij}{i+j-1} \times \int_L^U [m(x) x] x^{i+j-1} \, \mathrm{d}x - \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{1}{i!j!} \Phi^{(i)}(0) \Phi^{(j)}(0)^T \int_L^U m(x) x^{i+j} \, \mathrm{d}x \tag{6}$$

where the constant term evaluated at the limits of integration is identically zero for the pinned-free, fixed-free, and free-free cases by choice of the coordinate system.

Extracting terms from the second sum and combining the explicit sums, we may rewrite Eq. (6) as

$$[D] - [I] = \sum_{i=1}^{\infty} \sum_{j=2}^{\infty} \frac{1}{i!j!} \Phi^{(i)}(0) \Phi^{(j)}(0)^T \left[\frac{ij}{i+j-1} - 1 \right] \int_L^U m(x) x^{i+j} \, \mathrm{d}x + M \Phi(0) \Phi(0)^T \tag{7}$$

where M is the total mass of the beam. For the free-free case, we have used the fact that the eigenfunctions are orthogonal to any constant function; for the pinned-free and fixed-free cases, the same result—exclusive of the matrix $M\Phi(0)\Phi(0)^T$—is due to the boundary condition $\phi_i(0) = 0$. Letting ψ be the matrix of derivatives, $\psi(x)^T = [\Phi^{(2)}(x), \Phi^{(3)}(x), \Phi^{(4)}(x), \ldots]$, we can rewrite Eq. (7) as

$$[D] - [I] = M \Phi(0) \Phi(0)^T + \psi(0)^T [\tilde{M}] \psi(0) \tag{8}$$

where

$$[\tilde{M}]_{i,j} = \frac{ij}{(i+1)!(j+1)!(i+j+1)} \int_L^U m(x) x^{i+j+2} \, \mathrm{d}x \tag{9}$$

Since $M\Phi(0)\Phi(0)^T$ is positive semidefinite, this expression is positive definite if, and only if, the matrix $[\tilde{M}]$ is positive definite. We can rewrite this matrix as

$$[\tilde{M}] = \int_L^U [A]^T [H] [A] m(x) \, \mathrm{d}x \tag{10}$$

where $[A]_{i,j} = (ix^{i+1}) \delta_{ij} / (i+1)!$ and $[H]_{i,j} = 1/(i+j+1)$. The matrix $[H]$ is the Hilbert matrix, which is known to be positive definite. Hence, the matrix product $[A]^T[H][A]m(x)$ is positive definite for any positive weighting function $m(x)$,

Table 1 $[D]-[I]$ matrices for different mass profiles

		Uniform beam			
2.04976	0.00000	−0.58745	0.00000	−0.09087	0.00000
0.00000	5.39740	0.00000	−1.86112	0.00000	−0.36582
−0.58745	0.00000	10.32461	0.00000	−3.75429	0.00000
0.00000	−1.86112	0.00000	16.90500	0.00000	−6.19466
−0.09087	0.00000	−3.75429	0.00000	25.12963	0.00000
0.00000	−0.36582	0.00000	−6.19466	0.00000	34.99923
		Tapered beam			
1.33718	0.00000	−0.11555	0.00000	−0.16446	0.00000
0.00000	3.44538	0.00000	−0.63192	0.00000	−0.35044
−0.11555	0.00000	6.14599	0.00000	−1.14513	0.00000
0.00000	−0.63192	0.00000	9.83703	0.00000	−2.02919
−0.16446	0.00000	−1.14513	0.00000	14.70739	0.00000
0.00000	−0.35044	0.00000	−2.02919	0.00000	20.35255

Table 2 Eigenvalues of $[D]-[I]$ for different orders of approximation

Order	Eigenvalues					
			Uniform beam			
1	2.0498					
2	2.0498	5.3974				
3	2.0083	5.3974	10.3661			
4	2.0083	5.1039	10.3661	17.1985		
5	2.0024	5.1039	9.4743	17.1985	26.0273	
6	2.0024	5.0386	9.4743	15.3450	26.0273	36.9181
			Tapered beam			
1	1.3372					
2	1.3372	3.4454				
3	1.3344	3.4454	6.1488			
4	1.3344	3.3835	6.1488	9.8989		
5	1.3316	3.3835	5.9994	9.8989	14.8595	
6	1.3316	3.3653	5.9994	9.5361	14.8595	20.7336

and the integral of this matrix is also positive definite. Therefore, we conclude that, for a beam of arbitrary mass and stiffness distributions undergoing planar motion but not subject to external axial loading, centrifugal stiffening effects always dominate geometrical softening effects.

Numerical Example

As an illustrative example, we will consider two symmetric free-free beams of length of $2L = 144$ length units: a uniform beam and a linearly tapered beam. Letting x be measured from the center ($-72 \le x \le 72$), the respective mass profiles of these beams are $m(x) = M$ and $m(x) = M[0.2 + 1.6(L - |x|)/L]$, where the total mass of each beam is 0.144 units ($M = 0.001$). The stiffness distribution is analogous: M is replaced by $EI = 10{,}000$ in the preceding distributions. The principal mass moments of inertia of the beams about the axes of rotation are 249 and 241, respectively.

Table 1 shows the matrix $[D]-[I]$ for each of the cases, while Table 2 shows the eigenvalues of $[D]-[I]$ for different orders of system approximation up to sixth order. The fact that these are all positive values indicates that the matrices $[D]-[I]$ are positive for all orders of approximation. Note that the eigenvalues are roughly in proportion to the inertias. Since we would expect centrifugal effects to be greater for a beam with mass concentrated away from the center of mass, this result is quite reasonable.

Conclusions

It is shown that, for a rotating pinned-free or free-free beam confined to planar motion and subject to no external axial loading, the stiffening effect always dominates the softening. Although the proof may be valuable in itself, it is meant more as a template for further results. In the equations describing fully developed three-dimensional flexible and rigid body motion, stiffening is not guaranteed. This point will be addressed in a forthcoming paper.

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Linear Quadratic Regulator Approach to the Stabilization of Matched Uncertain Linear Systems

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Introduction

THE robust stabilization problem has received considerable attention in recent years. Nonlinear robust control laws that stabilize uncertain linear systems satisfying matching conditions were developed by Leitmann.[1] In addition, for matched uncertain linear systems, linear robust stabilizing control laws were developed by Thorp and Barmish,[2] Barmish et al.,[3] and Schmitendorf and Barmish.[4] However, the determination of these robust stabilizing control laws for matched uncertain systems involves complicated calculations. More recently, Jabbari and Schmitendorf[5] gave a noniterative method for design of linear robust stabilizing control laws based on a Lyapunov function approach. For mismatched uncertain linear systems, Schmitendorf[6] presented a scalar search procedure for determining a linear robust stabilizing controller. Moreover, Riccati equation approaches, which adjust a scalar to achieve the stabilization of systems with uncertainty parameters bounded by constraint sets, were derived by Petersen and Hollot[7] and Schmitendorf.[8]

This Note concentrates on the development of linear robust stabilizing control laws for matched uncertain linear systems

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